

Low Energy Effective Theory of QCD at High Isospin Chemical Potential

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The goal of this paper is to arrive at a low energy effective theory of QCD with two massless flavors of quarks at very high isospin density and zero baryon density. In a seminal paper by Son and Stephanov in the year 2001, it was conjectured that the low energy dynamics of QCD with two light flavors at asymptotically high isospin density was described by that of a pure Yang-Mills effective Lagrangian. Since the existence of a first order deconfinement phase transition with increasing temperature is a feature of every pure $SU(N)$ Yang-Mills theory with $N \geq 3$, the regime considered in this paper is also expected to exhibit a first order deconfinement phase transition with increasing temperature. However, the low energy constants (LEC) of this pure Yang-Mills theory have not been calculated till date. We calculate the LECs for this effective theory which in turn enables us to calculate the critical temperature of the deconfinement transition as a function of the isospin chemical potential μ_I . The scaling of the critical temperature as a function of the isospin chemical potential that we find in this paper can be compared with future lattice calculations at finite isospin density.

I. INTRODUCTION

One of the goals of modern nuclear physics is to understand the rich structure of the QCD phase diagram at finite density and temperature. This is because various such phases of matter can be realized in nature under different circumstances. The core of neutron stars and the evolution of the early universe are two of the most prominent examples where a knowledge of the phase diagram would help our understanding.

It is very difficult to get a handle on this problem starting directly from QCD. Accordingly it is useful to learn what we can from regimes which are tractable even when they are not of direct phenomenological relevance. In this paper we focus on the low temperature regime of two-flavor QCD with zero baryon chemical potential and very large isospin chemical potential. The study of this regime was originated by Son and Stephanov [1, 2] who noted that in the limit of extreme isospin chemical potentials this theory became equivalent to a pure Yang-Mills theory. This implies among other things that unlike at zero chemical potential the theory has a first-order deconfining transition. Although, the first order deconfinement transition at high isospin chemical potential was conjectured by Son and Stephanov, the critical temperature for this deconfinement transition has not been calculated before. This paper, relates the properties of this emergent Yang-Mills theory to the parameters of the underlying theory and calculates the critical temperature of this first order deconfinement transition.

One exciting feature of this analysis is that we can relate the equation of state at large isospin chemical potential and low temperature to the equation of state of pure Yang-Mills theory. Since the equation of state of

the pure Yang-Mills theory is computable using numerical studies on a lattice with relatively modest resources, this means that the equation of state of QCD at large isospin chemical potential and low temperatures is effectively computable with modest resources—including a computation of the critical temperature of the first order transition. The relation between the equations of state of QCD with two light flavors of quarks at finite isospin density and that of pure Yang-Mills is not presented here, but can be found in [3].

To put this problem in context, it is useful to recall that considerable progress has already been made in mapping the phase diagram both in isospin density vs temperature plane and baryon density vs temperature plane. The biggest hurdle impeding progress in this effort is the strength of the QCD coupling constant at moderate energy scales. A significant portion of the phase diagram which corresponds to observable phenomena such as the dynamics at the core of the neutron stars or the heavy ion collisions is inaccessible to perturbation theory calculations as the theory is strongly coupled in this regime. In order to circumvent this difficulty numerical calculations using lattice have been employed and this has been remarkably successful in some cases [4, 5]. But lattice QCD fails at finite baryon chemical potential at low temperature. The reason behind this is that the fermion determinant in the presence of a chemical potential for the quark number becomes complex, which causes lattice algorithms involving the method of important sampling to break down. This is known as the sign problem [6–9]. But an understanding of this regime of the phase diagram is absolutely necessary for neutron star physics. This

is because the mass-radius relation and the transport properties of a neutron star are determined by the phase of matter at the core where baryon density is high and the temperatures are low [10–14].

Apart from the regime of finite baryon density discussed in the previous paragraph, there exists another regime of the phase diagram, given by finite isospin chemical potential but zero baryon chemical potential. This paper explores the confinement dynamics in this regime. There are two reasons why this regime is of interest to us. The first reason is that in this regime lattice calculations are not plagued by the sign problem and hence may be practical. This means that lattice simulations can be used to test the results of model calculations and extrapolated results from rigorous perturbative calculations. In our paper we look at asymptotically high isospin chemical potential where we use perturbation theory to make predictions. Extrapolation of our results to moderate isospin densities, where perturbation theory breaks down can be compared with lattice results to determine the accuracy and the range of validity of the extrapolations. The second reason is that this regime may help give some insight into neutron stars as the imposition of charge neutrality in such objects (neutron star cores) leads to the suppression of proton fraction compared to the neutrons. This corresponds to a finite isospin chemical potential. Despite this, our system differs from the realistic scenario in these objects as there is also a finite baryon chemical potential in them as mentioned before.

Note that our system is unstable with respect to weak decay and is not electrically neutral. Hence the thermodynamic limit of such a system does not exist. But we can ignore the electromagnetic and weak effects in order to isolate the strong dynamics [2]. There have been considerable recent developments in this region of the phase diagram [1, 2, 15–21].

The phase structure of QCD at finite isospin chemical potential and zero baryon chemical potential with two light quarks was first considered in [1, 2]. For small enough μ_I , i.e. μ_I is much smaller than typical hadronic scales, chiral perturbation theory can be used to predict that charged pions (π^\pm) start condensing for $\mu_I > m_\pi$ where m_π is the mass of the pions. On the other hand, the opposite limit, where μ_I is much larger than typical hadronic scales and anti-up and down quarks form Fermi spheres of radius $+\frac{\mu_I}{2}$ respectively, interactions can be described using perturbative QCD due to asymptotic freedom. Just as in BCS in ordinary metals, if there is an instability at the Fermi surface caused by attractive interaction in some color, flavor and angular momentum channel, there will be a condensation of Cooper pairs in the problem at hand as well. In this high isospin regime, it is believed that, due to the strength of the attractive perturbative one gluon exchange interaction in the color singlet channel that a color singlet condensate of quark-antiquark Cooper pairs indeed does form. This means

that two possible forms of the condensate are $\bar{u}\gamma_5 d$ and $\bar{u}d$. The instanton induced interaction favours the condensation in $\bar{u}\gamma_5 d$ channel over the $\bar{u}d$ one and hence the condensate has pion quantum numbers. It was conjectured that there was a smooth crossover between pion superfluid at low isospin to quark-antiquark condensate at high isospin density [2]. For an asymptotically high μ_I the phase diagram in $\mu_I - T$ plane was argued to have a first order deconfinement phase transition apart from the second order superconducting phase transition at a higher temperature.

The argument is as follows. The condensate at high isospin density at temperatures smaller than the gap, has pion quantum numbers and is color neutral. This ensures that the gluons are not screened by the condensate. For temperatures well below the gap it is also energetically expensive to excite a quark quasi-particle out of the Fermi sphere and hence there is no Debye screening of the gluons either and we are left with a pure glue theory. A pure glue theory with $SU(3)$ gauge group is known [22] to have a first order deconfinement phase transition with increasing temperature. Therefore we can expect a similar behavior in our problem. It was argued by Son and Stephanov that the critical temperature of this deconfinement transition should be lower than the critical temperature of the superconducting transition [1]. Interestingly, as there is no phase transition with increasing temperature for $\mu_I = 0$, it was argued in [2] that the first order phase transition line has to end at a critical point if one assumes quark-hadron continuity along the $T = 0$ isospin axis.

As mentioned above, It is expected that for $\Delta \gg T$ and $\mu_I \gg m_\rho$ the static color charges are not screened due to the absence of both Debye and Meissner masses and the theory describing the low energy dynamics of the system is governed by the Lagrangian density $\mathcal{L} = \frac{-F^2}{4g^2}$ where g is the coupling constant that matches that of the original theory at the scale Δ . But this is not a complete picture. Although the quarks are bound in $SU(3)$ singlet Cooper pairs, they can still partially screen the static color charges by changing the chromodielectric (dielectric constant for the color electric field) constant of the theory ϵ from unity. As we will see, in this theory $\epsilon > 1$ and the coulomb potential between static color charges is smaller than $\frac{q^2}{r}$ by a factor of ϵ where r is the distance between the static charges. This amounts to having a pure glue theory where the speed of gluon is given by $\frac{1}{\sqrt{\epsilon\lambda}} < 1$ where, λ stands for the chromomagnetic permeability.

While this general picture was introduced some time ago [2] to the best of our knowledge, quantitative predictions of properties of the effective pure glue theory in terms of the underlying theory are lacking. A principal purpose of this paper is to compute this chromodielectric constant (and the chromomagnetic permeability) as a function of μ_I integrating out the quarks around the Fermi surface in an energy range of the order of the gap. For asymptotically high μ_I , we find that $\epsilon\lambda \gg 1$ where

ϵ and λ have been defined with respect to ϵ_0 and λ_0 of the vacuum by absorbing the vacuum polarization effect in g . We also predict the effective confinement scale as a function of μ_I . The confinement scale is related to the chromodielectric constant in a simple way which is also elaborated in this paper. The scaling of the confinement scale with μ_I obtained in this paper should be compared with scaling results obtained from lattice simulations if possible in future.

A particularly interesting fact is that there is a first-order deconfinement transition at large μ_I . We cannot analytically compute its critical temperature from first principles. However, if we can determine the ratio of the deconfinement transition temperature to the scale of the theory for pure Yang-Mills, we can then predict the phase transition temperature for large μ_I . Moreover, as noted earlier, the calculation of the deconfinement transition temperature for pure Yang-Mills is computable on the lattice with quite modest resources, so that a prediction of the QCD deconfinement phase transition at large μ_I is viable as is a more general calculation of the equation of state. The equation of state turns out to be different from the equation of state of pure Yang Mills theory suitably rescaled [3]. Apart from the low lying gluodynamics, the theory has a meson Goldstone mode which contributes to the equation of state. While the mode is massless, and thus it contributes at low energies, the scale of its interactions is well above scale of the gluodynamics. Thus, while the Goldstone mode contributes to the pressure, it is also effectively non-interacting and does not affect the gluodynamics.

The story of a pure glue theory anisotropic in space-time at sufficiently high isospin chemical potential is reminiscent of the ‘two flavor color-superconductor’ (2SC) phase at low temperature and high μ_B [23]. In case of the 2SC phase at finite baryon density, two quarks form a Cooper pair giving rise to a condensate that breaks color $SU(3)$ down to color $SU(2)$. Five of the eight gluons become massive due to the color-Meissner effect of the condensate, but the remaining three remain massless. At temperatures lower compared to the gap Debye screening for these three gluons is absent giving rise to pure $SU(2)$ gluodynamics. However, just like in our problem, the quark-loops alter the chromodielectric constant of the system.

The paper is organized as follows: Sec. 2 discusses the microscopic Lagrangian that is used to determine the low energy constants of the gluodynamics. Sec. III outlines the evaluation of the chromodielectric constant and the chromomagnetic permeability using the microscopic Lagrangian. Sec. IV estimates the energy scale of the deconfinement transition followed by a concluding section.

II. MICROSCOPIC LAGRANGIAN

In order to quantify the effect of the paired quarks on the chromodielectric constant of the system we need to

be able to write down the effective theory for the gluons. The low energy constants of this theory need to be extracted by integrating out quark loops from a microscopic Lagrangian, which in this case is the QCD Lagrangian. We emphasize again that since we are working at asymptotically high densities it is justified to make use of perturbation theory. We start with the QCD Lagrangian with two light (massless) quarks in the fundamental representation of the $SU(3)$ color group with an isospin chemical potential μ_I ,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu + \mu_I \gamma^0 \tau_3)\psi - \frac{1}{4}(F_{\mu\nu}^a)^2 \quad (1)$$

where,

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \quad (2)$$

, $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c$ and $D_\mu = \partial_\mu - ig A_\mu^a t^a$. The generators of color $SU(3)$ are denoted by t^a and τ_3 is the third Pauli matrix given by

$$\tau_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

u and d stand for up and down quarks respectively. As mentioned above the condensate has pion quantum numbers and is of the form $\bar{u}\gamma_5 d$. In order to analyse this condensate we define

$$\tilde{\psi} \equiv \begin{pmatrix} u \\ \tilde{d} \end{pmatrix} \quad (4)$$

where $\tilde{d} = \gamma_5 d$. We intend to find the propagator for the $\tilde{\psi}$ quarks in the presence of this condensate and to do so we introduce an auxiliary field Δ such that the inverse propagator for the quarks is given by

$$G^{-1} = -i \begin{pmatrix} \gamma^\mu p_\mu + \mu_I \gamma^0 & \Delta \\ \tilde{\Delta} & \gamma^\mu p_\mu - \mu_I \gamma^0 \end{pmatrix} \quad (5)$$

with $\tilde{\Delta} = \gamma_0 \Delta^\dagger \gamma_0$. The quarks are expected to acquire a gap in their spectrum which is to say that it is energetically expensive to excite a quark bound in a Cooper pair. This gap is characterized by the magnitude of the auxiliary field when the action is minimized with respect to the auxiliary field. This amounts to solving for the gap, Δ using Schwinger-Dyson (S-D) equation. We do not need to solve for Δ for the purpose of our paper, although it is straightforward to do so. The gap was estimated in [2] to be of the form $\Delta = b|\mu_I|g^{-5} \exp(-c/g)$ where $c = \frac{3\pi^2}{2}$ and g was evaluated at the scale μ_I . Note that, in ordinary superconductivity with a four-fermi interaction the gap is of the form $\Delta \sim \exp(-k/g^2)$ where k is some numerical constant. Here, however, $\Delta \sim \exp(-c/g)$. The source of this behavior is that instead of a contact four-fermi interaction we have long range magnetic interaction, just as in the case of a superconducting gap at large μ_B . b was estimated to be $\sim 10^4$ [2]. It is easy to see

from the expression for the gap that as g gets smaller the gap Δ becomes smaller compared to μ_I . This is a standard feature of any BCS calculation and is the only consistent solution of the gap equation other than $\Delta = 0$. In our problem as μ_I becomes large, the strength of the coupling becomes smaller and we obtain $\mu_I \gg \Delta$.

For our problem we can express the order parameter as

$$\tilde{\Delta} = i\Delta_4 + i\Delta_5 \hat{\gamma} \cdot \hat{k} \gamma_0 \quad (6)$$

$$= i(\Delta^+ \lambda_{\mathbf{p}}^+ + \Delta^- \lambda_{\mathbf{p}}^-) \quad (7)$$

. $\lambda_{\mathbf{p}}^+$ and $\lambda_{\mathbf{p}}^-$ are free quark on-shell projectors given by

$$\lambda_{\mathbf{p}}^\pm \equiv \frac{1}{2} (1 \pm \gamma_0 \boldsymbol{\gamma} \cdot \hat{\mathbf{p}}) \quad (8)$$

, where Δ^+ and Δ^- are real numbers. If we restrict ourselves to the 1S_0 channel, we have $\Delta^+ = \Delta^-$. For the purpose of this paper we are only interested in the 1S_0 channel and we do take $\Delta^+ = \Delta^- = \Delta$. With this structure of the condensate in mind, we return to deriving the propagator for the $\tilde{\psi}$ quarks. We do so by going to Euclidean time which corresponds to a finite temperature calculation and then take the limit $T \rightarrow 0$ at the end. The inverse quark-propagator in Euclidean action is given by

$$G^{-1} = \begin{pmatrix} -i\gamma^0 p^0 - \gamma^i p^i + \mu\gamma^0 & \Delta \\ \Delta & -i\gamma^0 p^0 - \gamma^i p^i - \mu\gamma^0 \end{pmatrix} \quad (9)$$

. Inverting G^{-1} we arrive at the propagator

$$G = \begin{pmatrix} G^+ & \Sigma^- \\ \Sigma^+ & G^- \end{pmatrix} \quad (10)$$

where

$$\begin{aligned} G^-(k) &= \sum_{e=\pm} \frac{-ik_0 + (\mu - ek)}{-k_0^2 - (\epsilon_k^e)^2} \lambda_{\mathbf{k}}^{-e} \gamma^0 \\ G^+(k) &= \sum_{e=\pm} \frac{-ik_0 - (\mu - ek)}{-k_0^2 - (\epsilon_k^e)^2} \lambda_{\mathbf{k}}^{+e} \gamma^0 \\ \Sigma^-(k) &= i \sum_{e=\pm} \frac{\Delta^e \lambda^e}{-k_0^2 - (\epsilon_k^e)^2} \\ \Sigma^+(k) &= -i \sum_{e=\pm} \frac{\Delta^e \lambda^{-e}}{-k_0^2 - (\epsilon_k^e)^2} \end{aligned} \quad (11)$$

and $(\epsilon_k^e)^2 = (|\mathbf{k}| - e\mu)^2 + (\Delta^e)^2$ where $e = \pm$. Note that, as expected, the dispersion relation of the quarks have become gapped. Also, the dispersion relations with $e = -1$ correspond to anti-up and down quarks where as the ones with $+1$ correspond to up and anti-down quarks. Apart from the propagator of the quarks in this basis, that is $\tilde{\psi}$ quarks, we also need the interactions of the $\tilde{\psi}$ quarks with the gluons to make any progress. The interaction term between the quarks and the gluons in the QCD Lagrangian, i. e. $\bar{\psi} \gamma^\mu A_\mu^a t^a \psi$ needs to be rewritten in the basis of the $\tilde{\psi}$ quarks. For the sake of clarity we write down explicitly the interaction term between the gauge field and $\tilde{\psi}$ quarks and it is given by

$$\bar{\tilde{\psi}} \begin{pmatrix} -ig\gamma^\mu A_\mu^a t^a & 0 \\ 0 & -ig\gamma^\mu A_\mu^a t^a \end{pmatrix} \tilde{\psi} \quad (12)$$

III. EFFECTIVE ACTION

Armed with the propagator for the quark and the interaction of the quarks with the gluons we can now write down the form of the low energy effective Lagrangian from symmetry arguments alone. The microscopic Lagrangian had $SU(3)_{color} \times SU(2)_{flavor} \times U(1)_{em}$ internal symmetries, none of which are broken by the condensate. This means that the low energy effective theory should respect these internal symmetries. The microscopic Lagrangian however, was not Lorentz invariant due to the presence of the chemical potential which broke Lorentz symmetry to rotational invariance explicitly. This means that the effective Lagrangian for the gluons will be rotationally invariant but not Lorentz invariant. Apart from this, by requiring locality the form of the effective action can be obtained as

$$S_{glue} = \sum_a \frac{1}{g^2} \int d^4q \left(\frac{\epsilon}{2} E_a^2 - \frac{1}{2\lambda} B_a^2 \right) \left(1 + O\left(\frac{q^2}{\Delta^2}\right) \right) \quad (13)$$

, where E^a and B^a are chromoelectric and chromomagnetic fields respectively, g is the QCD coupling at energy scale μ_I . For gluon momentum q , smaller than the gap Δ , higher dimensional operators which include powers of q or more powers of E and B fields are suppressed by powers of $\frac{|q|}{\Delta}$ and the effective action behaves like a pure Yang-Mills theory. ϵ and λ are the low energy constants, to be identified as the chromodielectric constant and the chromomagnetic permeability of the medium. These can be explicitly calculated by computing the polarization tensor for the gluons, which amounts to integrating out the quark quasiparticles upto energies of the order of the gap around the Fermi surface. In other words, we use the one loop polarization tensor to obtain the part of the effective action that is quadratic in the gluon fields and also contains the effects of the integrated out quarks. In this action for the gluons, we rewrite the gluon fields in terms of the chromoelectric and chromomagnetic fields using the definitions

$$E_i^a = F_{0i}^a \quad (14)$$

and

$$B_k^a = \epsilon_{ijk} F_{ij}^a \quad (15)$$

and compare with Eq. 13. The two Lagrangians are compared in order to write down ϵ and λ in terms of the various components of the polarization. Upto leading order in the expansion of gluon momentum over the gap, at zero temperature this matching calculation yields the following relations:

$$\begin{aligned} \Pi_{ab}^{00}(q_0, \mathbf{q}) &= -(\epsilon - 1) |\mathbf{q}|^2 \delta_{ab} \\ \Pi_{ab}^{ij} \frac{\delta^{ij}}{2} \frac{\delta_{ab}}{8} &= -(\epsilon - 1) q_0^2 \\ \Pi_{ab}^{ij}(q_0, \mathbf{q}) \left(-\delta^{ik} - \frac{q^i q^k}{|\mathbf{q}|^2} \right) &= \left(\frac{1}{\lambda} - 1 \right) |\mathbf{q}|^2 \delta_{ab} \delta^{jk} \end{aligned} \quad (16)$$

where, Π is the polarization of the gluons. The indices in superscripts on Π stand for space-time indices and the subscript for color indices.

We now have to calculate the polarization tensor. At one loop it is given by

$$\begin{aligned} \Pi_{ab}^{ij}(p) = g^2 T \sum_{k_0} \int \frac{d^3 k}{(2\pi)^3} \text{Trace} [& \gamma^i t_a G^+(k) \gamma^j t_b G^+(k-p) \\ & + \gamma^i t_a G^-(k) \gamma^j t_b G^-(k-p) \\ & + \gamma^i t_a \Sigma^-(k) \gamma^j t_b \Sigma^+(k-p) \\ & + \gamma^i t_a \Sigma^+(k) \gamma^j t_b \Sigma^-(k-p)] \end{aligned} \quad (17)$$

. The expressions for G^+ , G^- , Σ^+ and Σ^- are substituted from eq. 11 in eq. 17. It is easy to see after the substitution that $\Pi_{11}^{00} = \Pi_{22}^{00} = \Pi_{33}^{00}$ and so on.

At this point we should note that the trace in eq. 17 is a standard one and especially similar to the unbroken $SU(2)$ gluon polarization in the two flavor color-superconducting phase. The reason behind this similarity has been explained before and is related to the fact that the low energy dynamics of both the phases are driven by gluodynamics. The above mentioned trace was calculated by [24] and the integral over 3 momentum was computed in [23] in the context of the $2SC$ phase. We should remember that the three momentum integral is dominated by anti-up and down quarks with momenta close to the Fermi surface. Hence, the up and anti-down quark contribution to the above integral is neglected which amounts to neglecting terms with denominators $(|\mathbf{k}| - e\mu)^2 + (\Delta^e)^2$ or $(|\mathbf{k} - \mathbf{p}| - e\mu)^2 + (\Delta^e)^2$ with $e = -1$. Because the trace and the integral in eq. 17 are standard, instead of going into the details of the calculation we just state the results here. We find that as $T \rightarrow 0$, around $q_0, \mathbf{q} = 0$ up to leading order $\frac{|q|^2}{(\Delta^+)^2}$ expansion, $\Pi_{aa}^{00} = -\frac{g^2 \mu_I^2 |q|^2}{18\pi^2 (\Delta^+)^2}$ and $\Pi_{ab}^{ij} = \frac{g^2 \mu_I^2 q_0^2}{18\pi^2 (\Delta^+)^2} \delta^{ij} \delta_{ab}$. The appearance of μ_I in the numerator is due to the fact that the integral in eq. 17 is dominated by momenta near the Fermi surface. This means that the three momentum integral essentially turns into a one dimensional integral multiplied by the area of the Fermi surface which is proportional to the square of the Fermi momentum or the chemical potential. Note that $\Pi_{ab}^{ij}(-\delta^{jk} - \frac{q^i q^k}{|q|^2}) = 0$. Comparing eq. 16 with these expressions for Π_{aa}^{00} , Π_{ab}^{ij} we conclude that $\lambda = 1$ and $\epsilon = 1 + \frac{g^2 \mu_I^2}{18\pi^2 (\Delta^+)^2}$.

IV. CONFINEMENT

The effective Lagrangian that we obtained by integrating out the fermions has the form of a pure glue theory upto corrections $\sim \frac{|q|}{\Delta}$ and can be recast as

$$S = -\frac{1}{4(g')^2} \int d^4 x' (F')^2 \quad (18)$$

with some simple rescalings of the fields and the coupling etc. The action of 18 will however have a speed of gluon given by

$$c = \frac{1}{\sqrt{\lambda\epsilon}} = \frac{1}{\sqrt{\epsilon}} \quad (19)$$

$$\approx \frac{3\sqrt{2}\pi\Delta}{g\mu_I} \quad (20)$$

as $\lambda = 1$ and for $\mu_I \gg \Delta$. Before elaborating on the scaling, we would like to point out that g' , in eq. 18 runs exactly like the coupling in a pure $SU(3)$ Yang-Mills theory for momentums smaller than the gap. As the energy scales reach the gap from below and become bigger than the gap, the running behavior of the coupling will no longer be that of pure $SU(3)$ Yang-Mills and will get affected by the higher dimensional operators. Hence, the action of eq. 18 should be matched with the actions of eq. 13 at the scale Δ . To know how exactly this matching is done, we have to rescale time, fields and the coupling in eq. 13 as follows. We can rescale time direction, the gauge field and the coupling as $t_0'' = \frac{t_0}{\sqrt{\epsilon}}$, $A_0^{a''} = \sqrt{\epsilon} A_0^a$ and $g'' = \frac{g}{\epsilon^{1/4}}$ to rewrite the Lagrangian in eq. 13 as

$$S = -\frac{1}{4(g'')^2} \int d^4 x'' (F'')^2 \quad (21)$$

where $\alpha_s'' = \frac{\alpha_s(\mu)}{\sqrt{\epsilon}} \ll \alpha_s$. Now we have to match the coupling constant in the action 18, $\alpha_s'(q')$ with α_s'' at a scale $q' = \Delta$ which in turn means

$$\alpha_s'(\Delta) = \alpha_s'' = \frac{\alpha_s(\mu)}{\sqrt{\epsilon}} \ll \alpha_s. \quad (22)$$

With this pure Yang-Mills theory in hand, we now concentrate on the dynamics of the deconfinement transition. As mentioned in the introduction, a pure $SU(3)$ Yang-Mills theory undergoes a first order deconfinement phase transition with increasing temperature. This means that in our problem at finite temperature, for temperatures smaller compared to the gap there will be a deconfinement phase transition with increasing temperature. The energy scale of this deconfinement transition will be much smaller than the gap and we call this new confinement scale $\tilde{\lambda}$ to distinguish it from the zero density QCD confinement scale λ_{QCD} . We can figure out how the confinement scale of the effective theory compares with the gap by looking at the renormalization group equation

$$\begin{aligned} \tilde{\lambda} &= \Delta \exp \left(-\frac{2\pi}{b_0 \alpha_s'(\Delta)} \right) \\ &= \Delta \text{Exp} \left[-2\sqrt{2\pi} \frac{\mu}{29\Delta \sqrt{\alpha_s'(\Delta)}} \right] \end{aligned} \quad (23)$$

where b_0 at one loop with two flavours for color $SU(3)$ is given by $b_0 = 11 - \frac{2}{3}n_f = \frac{29}{3}$ and α_s' is the coupling evaluated at the scale Δ . To evaluate the confinement

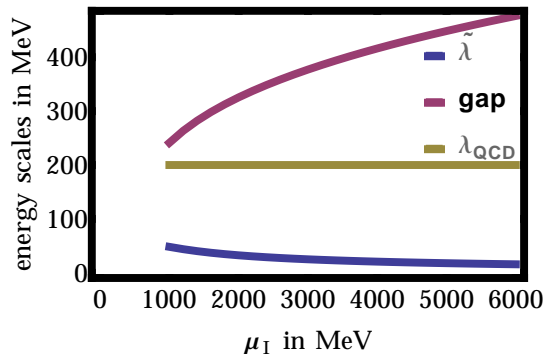


FIG. 1: Confinement scale in MeV as a function of chemical potential in MeV

scale we have to evaluate the gap. The dependence of Δ on the coupling is

$$\Delta^+ = b|\mu_I|g(\mu_I)^{-5} \exp\left(-\frac{3\pi^2}{2g(\mu_I)}\right) \quad (24)$$

with $b = 10^4$ [1]. We compute and plot the confinement scale for a range of the isospin chemical potential from asymptotically large values where we trust our calculations completely to regions of smaller values where the approximations start breaking down. In the region of smaller isospin density the confinement scale $\tilde{\lambda}$ could be appreciably different from what we found here. It would be instructive to compare our results with a future lattice calculation in the regime of small isospin density ≤ 1000 MeV where the different scales in the theory are not so well separated. Fig. 1 illustrates the various energy scales in the problem as a function of μ_I . It also demonstrates how the scales get widely separated for asymptotically high isospin density where perturbation theory is valid and how they get closer in the opposite limit with low isospin density where the QCD coupling becomes considerably strong. For example if we look at fig. 1 for $\mu_I \sim 5000$ MeV, we notice that $\Delta \sim 400$ MeV and this is an indication of the fact that our calculations are trustworthy in this regime as the gap is much smaller than the isospin chemical potential. As expected from the eq. 23 the confinement scale decreases exponentially with the isospin chemical potential and this is illustrated in fig. 2.

V. CONCLUSION

Although analytical calculations presented here are only reliable in the high μ_I and low T region, one can deduce a few qualitative features about the phase diagram from these. Several features of the phase diagram should be regarded as being on solid ground. The first is that at $\mu_I = 0$, there are no phase transitions as T changes. The second is that for sufficiently high μ_I there exists a superconducting phase with a gap serving as an order parameter. At low temperatures and isospin densities this

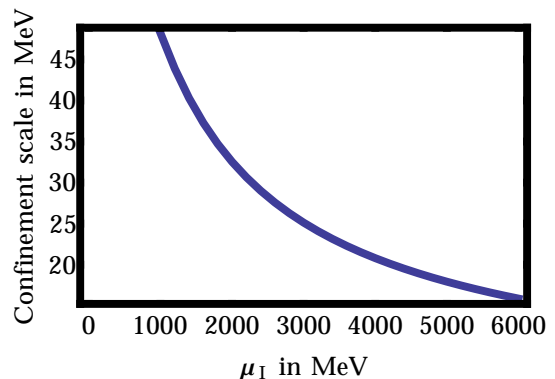
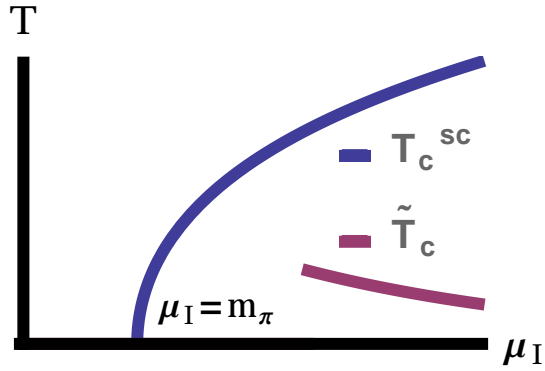


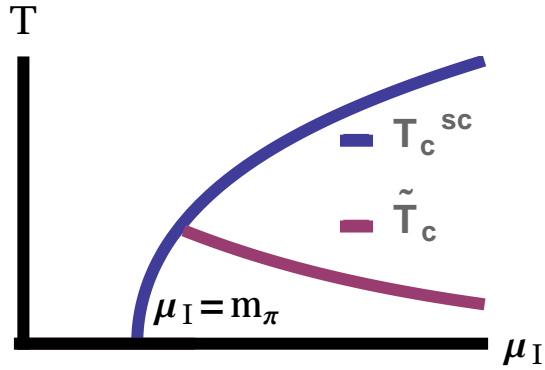
FIG. 2: The QCD confinement scale, the confinement scale at high isospin and the gap are shown in the same plot. The regime towards the right with higher isospin is where all these scales are well separated and we trust our calculations. The scales get closer as we go towards the left of the plot where the isospin chemical potential is small. This is the regime where a lattice calculation could determine how accurate the predictions are.

superconducting phase may be regarded as a pion condensate. At high temperature however this condensate melts. Thus there must exist a curve which separates the superconducting phase from the normal phase. The third robust feature of the phase diagram was the focus of this paper—namely the existence of a first-order phase transition at sufficiently high μ_I between a confining and a deconfining phase. This implies the existence of a curve which separates the confined phase from the deconfined phase.

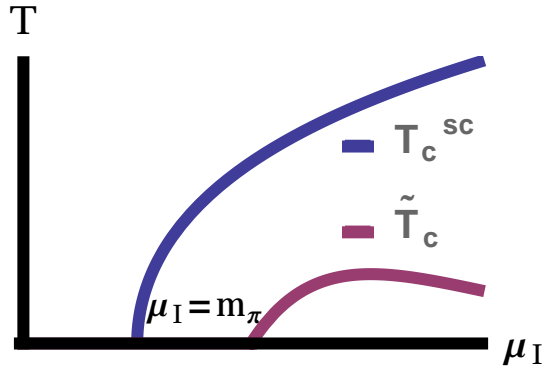
These three facts imply the existence of certain special points in the phase diagram. This is because, as was pointed out in [1] we also know that curves associated with the two transitions (superconducting and deconfinement) must end somewhere at sufficiently low μ_I , since there are no transitions at $\mu_I = 0$. The points where the curves end are special. For the case of the superconducting transition, this special point occurs at $T = 0$ and $\mu_I = m_\pi$ as is seen in all three curves in Fig. 3. An interesting and nontrivial fact is that one can infer the existence of an additional special point where the deconfinement transition ends. Son and Stephanov speculated about the nature of this point [1]. They argued that it is very plausible that there is no phase transition at zero T as one increases μ_I from m_π (where the superconducting transition occurs) as there is no symmetry. A very natural way for this to occur is the one suggested in [1] and shown in Fig. 3a, namely for the curve of first-order transitions separating the confined from deconfined phases to end at a critical point. This possibility is particularly exciting since critical points are characterized by universal behavior. However, we should note that it is not guaranteed to occur. For example, one could imagine that the special point associated with the end of the confinement transition could be a triple point as shown in Fig. 3b



(a)



(b)



(c)

FIG. 3: The three possible scenarios describing what happens to the deconfinement phase transition line as μ_I is lowered. T_c^{sc} corresponds to the critical temperature for the superconducting transition and \tilde{T}_c is the critical temperature for the deconfinement transition.

Alternatively, one might imagine, that despite the plausibility argument in ref. [1], that the curve might end at $T = 0$ as in Fig. 3c indicating a second transition at $T = 0$. One can also imagine more baroque scenarios with additional phases. The point, though is that however, the curve ends, it must end somewhere and the point at which this happens is intrinsically of interest. Ultimately, one would hope that lattice calculations can determine the phase structure and the position of these interesting points.

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